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Roughness at the depinning threshold for a long-range elastic string

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In this paper, we compute to high precision the roughness exponent ζ of a long-range elastic string, at the depinning threshold, in a random medium. Our numerical method exploits the analytic structure of the problem (''no-passing'' theorem), but avoids direct simulation of the evolution equations. The roughness exponent has recently been studied by simulations, functional renormalization-group calculations, and by experiments (fracture of solids, liquid meniscus in ^{4}He). Our result $\zeta = 0.388 \pm 0.002$ is significantly larger than what was stated in previous simulations, which were consistent with a one-loop renormalization-group calculation. Furthermore, the data are incompatible with the experimental results for crack propagation in solids and for a ^{4}He contact line on a rough substrate. This implies that the experiments cannot be described by pure harmonic long-range elasticity in the quasistatic limit.

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The statics and dynamics of elastic manifolds in random media govern the physics of a variety of systems, ranging from vortices in type-II superconductors [1] and charge density waves [2] to interfaces in disordered magnets [3], contact lines of liquid menisci on a rough substrate [4], and to the propagation of cracks in solids [5].

In most cases, the restoring elastic forces acting on a point on the manifold are local, i.e., depend only on the deformation in its neighborhood. The corresponding short-range string has been the object of many theoretical and experimental studies. In the depinning limit, two different scenarios are possible: numerical simulations and analytical calculations [6–8] have established that a string with an elastic restoring force breaks at the depinning threshold, while percolation experiments and numerical studies on directed polymers in random media [9,10] agree that in those systems with stronger than harmonic restoring forces the roughness exponent is $\zeta = 0.63$.

It has also been shown [5,11] that for a contact line of a liquid meniscus or for crack propagation in a solid, the elastic force is long range, rather than local. Nonlocal elasticity can be expected to modify the dynamic and static properties of these systems and to change the critical exponents. In this paper, we compute one of these exponents, the roughness exponent ζ of a long-range elastic string at the depinning threshold f_c.

The theoretical approaches are up to now based on the assumption that the motion of the line at the threshold is quasistatic. This means that velocity-dependent terms in the equation of motion of the deformation field $h(x,t)$ are taken to be irrelevant and that it can be derived from an energy function which incorporates potential energy due to the driving force f and the disorder potential $\eta(x,h)$, as well as elastic energy. According to this hypothesis, the equation of motion of the deformation field at zero temperature is

$$\frac{\partial}{\partial t} h(x,t) = f + \eta(x,h) - k \int dx_1 \frac{h(x,t) - h(x_1,t)}{(x-x_1)^2}. \quad (1)$$

The last term in this equation accounts for long-range restoring forces. Let us note that measurements of local velocities for a liquid ^{4}He contact line [4] have cast doubts on the validity of the quasistatic hypothesis for the present experiments.

The critical behavior of Eq. (1), at the driving force f equal to f_c, was studied by means of renormalization-group (RG) techniques. The one-loop calculations [12] gave a roughness exponent $\zeta^{(1)}$ equal to 1/3, which at a time was believed to be exact [12,13]. Early simulations based on extremal long-range models [14,15] ($\zeta = 0.35 \pm 0.02$) and on cellular automata [16] ($\zeta = 0.34 \pm 0.02$) found good agreement with this conjecture. However, experiments, both in crack propagation [17] and for a liquid ^{4}He contact line on a rough substrate [4] have measured, near f_c, a systematically larger exponent ($\zeta = 0.56 \pm 0.03$). Chauve *et al.* [8] recently showed that higher-order terms contribute to the RG result. At two-loop order, they found an exponent $\zeta^{(2)} = 0.47$. The large difference of the two-loop calculation with the lowest-order result let it seem conceivable that the experimental value could be explained by the model equation (1). In fact, the authors of Ref. [8] estimated the exponent to be $\zeta = 0.5 \pm 0.1$, which did include the experimental results.

In this paper we present a very precise method to determine the roughness exponent at f_c. We show that the exponent ζ has the value 0.388 ± 0.002. ζ is thus bigger than what was suggested by earlier simulations (cf., however, [18]) but is, at the same time, incompatible with the experimental situation.

The direct numerical simulation of the dynamics, especially of long-range systems, in the depinning region, is extremely tedious because the velocity of the manifold vanishes at the threshold, which is thus difficult to approach [19,20]. Many authors, therefore, preferred to treat the problem within the framework of cellular automaton models. These approaches can be very useful, but are hardly identifiable with a continuum equation (concerning the long-range case; cf. Refs. [16,21]).

However, there is an additional analytical structure in this problem, as first noticed by Middleton [23] for the continuum models. For short-range lattice models [10,22], we used this additional information to compute the critical string

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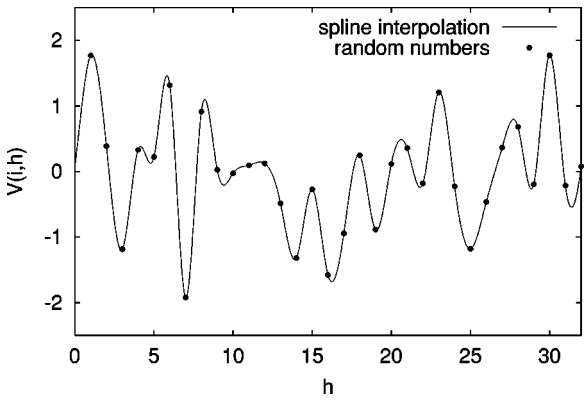

FIG. 1. Example of a random potential on the ith site with periodic boundary conditions ($M=32$). The circles are Gaussian random numbers with zero mean and unit variance, assigned to evenly spaced points $h_i=1,2,\ldots,M$. These numbers are then interpolated by a periodic cubic spline, in order to yield a continuous random potential $V(i,h_i)$.

(the blocked string at f_c) in an extremely efficient way, but without actually simulating the time evolution of the system. On the lattice, the long-range model does not seem to be open to such an approach, but the continuum model is, as we will show in this paper.

To proceed, we discretize the variable $x(x\to x_i, i=1,\ldots,L)$ in Eq. (1) and write h_i^t instead of $h(x_i,t)$. The complete string at time t is thus given by $h^t=\{h_1^t,h_2^t,\ldots,h_L^t\}$, where the h_i^t are real periodic variables for which h_i+M and h_i are identified. Periodic boundary conditions are also applied in L: $h_{L+i}=h_i$. The equation of motion (1) is adapted to accommodate the periodic boundary conditions. We have computed the long-range force $f^{\rm el}$ by summing over periodic images, as others have done before us [14,15], obtaining

$$f^{\rm el}[h_i]=\sum_{i_1=1}^{L-1}\left[2\psi'(i_1/L)-\pi^2/\sin^2(\pi i_1/L)\right]\frac{h_i^t-h_{i_1}^t}{L^2}, \tag{2}$$

where $\psi=d\Gamma(x)/dx$ and $\Gamma(x)$ is the gamma function. Different calculations, without the sum of images, gave an identical result for ζ. A random potential $V(i,h_i)$ acts on each site of the string. Our choice of the random potential is shown in Fig. 1. It allows to obtain a differentiable potential with $\eta(i,h_i)=-\partial V(i,h_i)/\partial h_i$.

The ''no-passing'' rule [23] establishes the following: if two strings h and $\tilde{h}$ do not cross at a given time t (say, $h_i^t<\tilde{h}_i^t\ \forall i$), they will not cross at any later time. Another important property of Middleton's theorem states that if, at an initial time $t_{\rm init}$, the velocities v_i^t are non-negative for all i, they will remain so for all later times $t>t_{\rm init}$. It follows from this property that, once we have found a forward moving string $h^{t_{\rm init}}$, we can be sure that snapshots of the string at later times will never cross. In fact, the strings h^t for $t>t_{\rm init}$ will form a noncrossing family with non-negative velocities, which satisfies the following conditions:

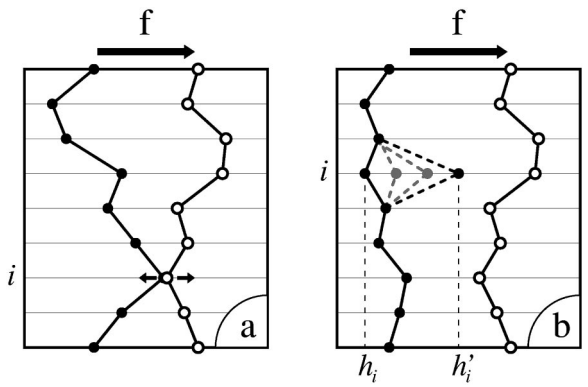

FIG. 2. (a) No-passing theorem: h^{t^*} (filled circles) and $h^{\rm block}$ (open circles). If h^{t^*} approaches $h^{\rm block}$ in a point i, the disorder forces become equal, but the elastic term (represented by arrows) prevents h^t from exceeding $h^{\rm block}$ for $t>t^*$. (b) Algorithm. The point i of the string (filled circles) is moved from h_i to h_i' in one step. Between h_i and h_i', the string's velocity in i remains positive and is zero at h_i'. A blocked configuration is drawn (open circles): our string may approach this configuration, but cannot pass it.

$$\begin{aligned}
&\text{(i)}\quad h_i^t\geqslant h_i^{t'}\ \ \forall i\ \text{ for }\ t>t'>t_{\rm init},\\
&\text{(ii)}\quad h_i^t\to h_i^{t'}\ \text{ for }\ t>t'>t_{\rm init},\quad t\to t',\\
&\text{(iii)}\quad v(h_i^t)\geqslant 0\ \ \forall i\ \text{ for }\ t>t_{\rm init},\\
&\text{(iv)}\quad \frac{\partial}{\partial t}h_i^t=v(h_i^t)\ \text{ from Eq. (1)}.
\end{aligned} \tag{3}$$

The velocity $v(h_i^t)$ is given, as usual, by $v(h_i^t)=-\partial E[h^t]/\partial h_i^t$ where $E[h^t]$ is the energy of the configuration h^t. In this context, we have made the following observation: if our only aim is to obtain the critical string, rather than to simulate the true time behavior, it is sufficient to generate continuous noncrossing families satisfying (i), (ii), and (iii), without imposing (iv). In this case, t would not be the physical time, but simply an ordering index; only then are the three conditions (i), (ii), and (iii) independent.

To assure that we can drop the condition (iv), we have to guarantee that no member of such a family will ever cross a blocked string $h^{\rm block}$ [by definition $v(h_i^{\rm block})\equiv 0$], if $h^{t_{\rm init}}$ did not. Let us suppose the contrary: if h^{t^*} were the first member which touches the blocked line in one point i [as shown in Fig. 2(a)], the random force for $h_i^{\rm block}$ and $h_i^{t^*}$ would become equal, but the elastic term would give $v(h_i^{t^*})<v(h^{\rm block})$. This would be a violation of condition (iii). Conversely, it is easy to see that such a family can always be continued up to a blocked line, because it suffices to find a single point i with positive velocity to continue the construction.

In previous works [19], the critical line was computed by direct simulation of Eq. (1). Notice that the discretization of time can pose difficult problems; during the interval time Δt the motion does not respect Middleton's theorem and, therefore, is not guaranteed to halt in front of a blocked string.

The construction of continuous noncrossing families of strings presents a much more powerful, completely rigorous approach. As indicated in Fig. 2, we consider in practice families in which at a given instant only a single coordinate i moves. Coordinate i is then advanced (from h_i to h_i') until its velocity vanishes [$v(h_i')=0$]. This is of course not done by simulation, but in a single step, by computing the zeros of $v(h_i)$. To simplify our numerical work, we have represented the continuous random potential as a spline (a piecewise third-order polynomial in h), as further explained in Fig. (1). This allows us to solve for the closest zero of the velocity function from a quadratic equation. The only parameter in this procedure is the minimal velocity below which the *total* string is assumed blocked, and which serves to terminate our iteration. We have varied this nonessential parameter by four orders of magnitude, and shown that the critical forces and exponents are extremely well stabilized.

An initial forward-moving string $h^{t_{\rm init}}$ is very easy to obtain. We have remarked that our iterative algorithm, which is completely rigorous, converges extremely quickly to a blocked string if such an object exists. A bisection method then allows us to obtain the critical driving force f_c, and the critical string. We stress again that, by construction, a blocked string is never passed.

We have run this algorithm on a large number of samples with $L=M$. Sample sizes varied from $L=8$ to $L=2048$. We compute the mean-square elongation $W^2(L)$ of a critical string h^c as

$$W^2(L) := \overline{\langle (h^c - \langle h^c \rangle)^2 \rangle} \tag{4}$$

In Eq. (4), $\langle h^c \rangle = 1/L \Sigma_i h_i^c$, while the overbar stands for an average over the disorder. Our data extrapolate very well and a mean-square analysis including the data from $L=L_{\rm min}=16$ up to $L=2048$ yields $\zeta=0.388\pm0.002$. In the inset of the figure, we show that this mean-square analysis does not systematically depend on $L_{\rm min}$, the smallest sample size included in our fit, for sizes larger than $L_{\rm min}=8$ (as can be seen in Fig. 3, the $L=8$ data are too low).

We believe that the linear scaling $(L,M)\rightarrow(\alpha L,\alpha M)$ [22] is the correct way to approach the thermodynamic limit in this system, for which $\zeta<1$. This has also been assumed in all previous work on this and related systems [16,19,20]. We have explicitly checked that the result for the roughness exponent does not depend on the ratio L/M, as is evident [22]. More importantly, one should expect identical results even for scalings $M\sim L^{\zeta'}$ for $\zeta'>\zeta$, as the critical line only correlates sites on a scale L^{ζ}. This is indeed what we find.

As an example, we show in Fig. 3 data for a scaling $M=L^{1/2}$. For these systems, the critical force is smaller and the extension of the critical line is smaller, but the roughness exponent is unchanged, lending further credit to our value of the roughness exponent, and to the tight error bar. Too small values of M cannot reasonably be studied on a finite system, because the critical line may then wind around the periodic box, introducing spurious effects, which are difficult to include into a finite-size analysis. It is for this reason that we have not studied values of ζ' very close, or identical to ζ.

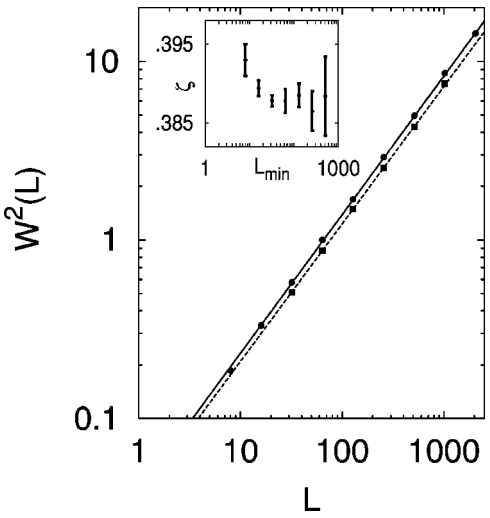

FIG. 3. Mean-square elongation $W^2(L)$ as a function of system size L for the long-range elastic force given by Eq. (1). Upper (full) line: linear scaling $M=L$ for system sizes between $L=8$ and $L=2048$. The interpolating line corresponds to a roughness exponent $\zeta=0.388\pm0.002$. Inset: roughness exponents ζ obtained for system sizes $L\geqslant L_{\rm min}$ vs $L_{\rm min}$. Lower (broken) line: sublinear scaling $M=L^{1/2}$ yields identical values for the roughness exponent.

In conclusion, we have obtained the roughness exponent for long-range elastic strings by a numerical procedure which respects the analytic structure in the problem (no-passing theorem) and allows to obtain very high precision. Our approach can certainly be extended to other problems, and we would in particular be interested in simulation methods which allowed to study, e.g., the KPZ equation [24] directly in the continuum. This is most important, as it has been shown that the effect of discretizations can be very difficult to control [25].

The large difference of the result with the experimental value shows that the theoretical model of these processes will certainly have to be modified in an essential way. One possibility is that velocity-dependent terms have to be taken into account. This appears reasonable, as large local velocities have been observed in the experiments.

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